Advanced Mathematics

An Incremental Development

SECOND EDITION

SAXON

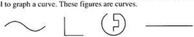
Contents

LESSON 1 Geometry Review

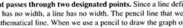
1.A points, lines, and rays

Some fundamental mathematical terms are impossible to define exactly. We call these terms **primitive terms** or **undefined terms.** We define these terms as best we can and then use them to define other terms. The words **point, curve, line,** and **plane** are primitive terms.

A point is a location. When we put a dot on a piece of paper to mark a location, the dot is not the point because a mathematical point has no size and the dot does have size. We say that the dot is the **graph** of the mathematical point and marks the location of the point. A curve is an unbroken connection of points. Since points have no size, they cannot really be connected. Thus, we prefer to say that a curve defines the path traveled by a moving point. We can use a pencil to graph a curve. These figures are curves.



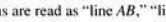
A mathematical line is a straight curve that has no ends. **Only one mathematical line can be drawn that passes through two designated points.** Since a line defines the path of a moving point that has no width, a line has no width. The pencil line that we draw marks the location of the mathematical line. When we use a pencil to draw the graph of a mathematical line, we often put arrowheads on the ends of the pencil line to emphasize that the mathematical line has no ends.



We can name a line by using a single letter (in this case, p) or by naming any two points on the line in any order. The line above can be called line AB, line BA, line AM, line MA, line BM, or line MB. Instead of writing the word *line*, a commonly used method is to write the letters for any two points on the line in any order and to use an overbar with two arrowheads to indicate that the line continues without end in both directions. All of the following notations name the line shown above. These notations are read as "line AB," "line BA," etc.

$$\overleftrightarrow{AB} \qquad \overleftrightarrow{BA} \qquad \overleftrightarrow{AM} \qquad \overleftrightarrow{MA} \qquad \overleftrightarrow{BM} \qquad \overleftrightarrow{MB}$$

We remember that a part of a line is called a **line segment** or just a **segment.** A line segment contains the endpoints and all points between the endpoints. A segment can be named by naming the two endpoints in any order. The following segment can be called segment AB or segment BA.



Instead of writing the word *segment*, we can use two letters in any order and an overbar with no arrowheads to name the line segment whose endpoints are the two given points. Therefore, $\overline{AB}$ means "segment AB" and $\overline{BA}$ means "segment BA." Thus, we can use either

$$\overline{AB} \quad \text{or} \quad \overline{BA}$$

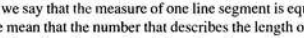
to name a line segment whose endpoints are A and B.

The length of a line segment is designated by using letters without the overbar. Therefore, AB designates the length of segment AB and BA designates the length of segment BA. Thus, we can use either

$$AB \quad \text{or} \quad BA$$

to designate the length of the line segment shown below whose endpoints are A and B.

The words **equal to, greater than,** and **less than** are used only to compare numbers. Thus, when we say that the measure of one line segment is equal to the measure of another line segment, we mean that the number that describes the length of one line segment is equal to the number that describes the length of the other line segment. Mathematicians use the word **congruent** to indicate that designated geometric qualities are equal. In the case of line segments, the designated quality is understood to be the length. Thus, if the segments shown here

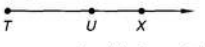


are of equal length, we could so state by writing that segment PQ is congruent to segment RS or by writing that the length of $\overline{PQ}$ equals the length of $\overline{RS}$. We use an equals sign topped by a wavy line (≅) to indicate congruence.

Congruence of Line Segments		Equality of Lengths
$\overline{PQ} \cong \overline{RS}$	or we could write	$PQ = RS$

Sometimes we will use the word *congruent* and at other times we will speak of line segments whose *measures* are equal.

A **ray** is sometimes called a **half line.** A ray is part of a line with one endpoint–the beginning point, called the **origin**–and extends indefinitely in one direction. The ray shown here begins at point T, goes through points U and X, and continues without end.

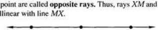


When we name a ray, we must name the origin first and then name any other point on the ray. Thus we can name the ray above by writing either "ray TU" or "ray TX." Instead of writing the word *ray*, we can use two letters and a single-arrowhead overbar. The first letter must be the endpoint or origin, and the other letter can be any other point on the ray. Thus, we can name the ray shown above by writing either

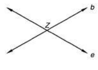
$$\overrightarrow{TU} \quad \text{or} \quad \overrightarrow{TX}$$

These notations are read as "ray TU" and "ray TX."

Two rays of opposite directions that lie on the same line (rays that are **collinear**) and that share a common endpoint are called **opposite rays.** Thus, rays XM and XP are opposite rays, and they are both collinear with line MX.

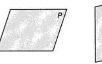


If two geometric figures have points in common, we say that these points are points of **intersection** of the figures. We say that the figures intersect each other at these points. If two different lines lie in the same plane and are not parallel, then they intersect in exactly one point. Here we show lines b and e that intersect at point Z.



1.B planes

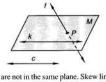
A mathematical line has no width and continues without end in both directions. A mathematical plane can be thought of as a flat surface like a tabletop that has no thickness and that continues without limit in the two dimensions that define the plane. Although a plane has no edges, we often picture a plane by using a four-sided figure. The figures below are typical of how we draw planes. We label and refer to them as plane P and plane Q, respectively.



Just as two points determine a line, three noncollinear points determine a plane. As three noncollinear points also determine two intersecting straight lines, we can see that two lines that intersect at one point also determine a plane.

On the right, we see that two parallel lines also determine a plane. We say that lines that lie in the same plane are **coplanar.**

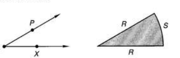
A line not in a plane is parallel to the plane if the line does not intersect the plane. If a line is not parallel to a plane, the line will intersect the plane and will do so at only one point. Here we show plane M and line k that lies in the plane. We also show line c that is parallel to the plane and line f that intersects the plane at point P.



Skew lines are lines that are not in the same plane. Skew lines are never parallel, and they do not intersect. However, saying this is not necessary because if lines are parallel or intersect, they are in the same plane. Thus, lines k and f in the diagram above are skew lines because they are not both in plane M, and they do not form another plane because they are not parallel and they do not intersect.

1.C angles

There is more than one way to define an angle. An angle can be defined to be the **geometric figure** formed by two rays that have a common endpoint. This definition says that the angle is the set of points that forms the rays, and that the measure of the angle is the measure of the opening between the rays. A second definition is that the angle is the **region** bounded by two radii and the arc of a circle. In this definition, the measure of the angle is the ratio of the length of the arc to the length of the radius.

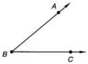


1. Three times the complement of angle A is 60° less than the supplement of angle A. Find the measure of angle A.

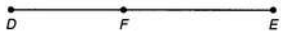
2. The ratio of students to teachers in the school was 8 to 5. If there were 1400 students, how many teachers were there?

3. In a taste test, 72% of the people polled preferred cereal B. If a total of 936 people polled preferred cereal B, how many people were polled?

4. Construct an angle which is congruent to $\angle ABC$, then bisect it.



5. Construct a perpendicular to $\overline{DE}$ at F.



6. Construct a triangle whose sides have lengths a, b, and c.

7. Solve: $\begin{cases} 2x + 6y = -36 \\ x - 3y = 0 \end{cases}$

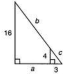
8. Solve: $6(x + x^0 - 1) = 2(-x + 8)$

9. Add: $\dfrac{5}{x(x+1)} + \dfrac{4}{x+1} + \dfrac{3}{x}$

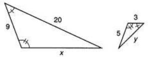
10. Expand: $\dfrac{4p^3s^{-5}}{p^{-3}s}\left(\dfrac{2^{-1}p^{-2}s}{p^3} + \dfrac{p^2s^3}{s^{-3}}\right)$

11. The lengths of the sides of a triangle are 8 m, 6 m, and 5 m. Is the triangle a right triangle, an acute triangle, or an obtuse triangle?

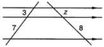
12. Find a, b, and c.



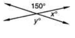
13. Find x and y.



14. Find z.



15. Find x and y.



16. An equilateral triangle has a perimeter of 24 cm and an area of $16\sqrt{3}$ cm^2. Find the altitude of the triangle.

17. A sphere has a radius of 5 feet. Find the volume and surface area of the sphere.

18. In the circle, O is the center. The radius of the circle is $\sqrt{8}$ meters. Find the area of the shaded sectors.

19. Find the volume of the cone whose base is shown and whose altitude is 8 cm. Dimensions are in centimeters.

20. Evaluate: $x^3 - 3y^3 + 2(x - y)(x^2 + 3xy + y^2)^0$ if $x = 3$ and $y = 2$

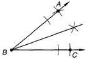
Test 1

1. (1)
$$3(90 - A) = (180 - A) - 60$$
$$270 - 3A = 120 - A$$
$$2A = 150$$
$$A = \mathbf{75°}$$

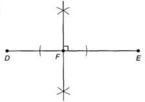
2. (R)
$$\frac{s}{t} = \frac{8}{5}$$
$$\frac{1400}{t} = \frac{8}{5}$$
$$8t = 7000$$
$$t = \mathbf{875}$$

3. (R)
$$\frac{72}{100} = \frac{936}{T}$$
$$72T = 93{,}600$$
$$T = \mathbf{1300}$$

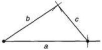
4. (4)



5. (4)



6. (4)



7. (R)
$$x - 3y = 0$$
$$x = 3y$$
$$2x + 6y = -36$$
$$2(3y) + 6y = -36$$
$$12y = -36$$
$$\mathbf{y = -3}$$
$$x = 3y$$
$$x = 3(-3)$$
$$\mathbf{x = -9}$$

8. (R)
$$6(x + x^0 - 1) = 2(-x + 8)$$
$$6x + 6 - 6 = -2x + 16$$
$$8x = 16$$
$$x = \mathbf{2}$$

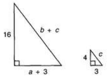
9. (R)
$$\frac{5}{x(x+1)} + \frac{4}{(x+1)} + \frac{3}{x}$$
$$= \frac{5 + 4x + 3(x+1)}{x(x+1)} = \mathbf{\frac{7x + 8}{x^2 + x}}$$

10. (R)
$$\frac{4p^3s^{-5}}{p^{-3}s}\left(\frac{2^{-1}p^{-2}s}{p^3} + \frac{p^2s^3}{s^{-3}}\right)$$
$$= \frac{4p^6}{s^6}\left(\frac{s}{2p^5} + p^2s^6\right) = \mathbf{\frac{2p}{s^5} + 4p^8}$$

11. (3)
$$c^2 \bigcirc a^2 + b^2$$
$$8^2 \bigcirc 6^2 + 5^2$$
$$64 \bigcirc 36 + 25$$
$$64 > 61$$

Since the square of the largest side is greater than the sum of the squares of the other two sides, the triangle is an **obtuse triangle.**

12. (3)

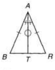


$$c^2 = 4^2 + 3^2$$
$$c = \sqrt{16 + 9}$$
$$\mathbf{c = 5}$$

$$\frac{4}{16} = \frac{3}{3 + a}$$
$$12 + 4a = 48$$
$$4a = 36$$
$$\mathbf{a = 9}$$

$$\frac{4}{16} = \frac{c}{b + c}$$
$$\frac{4}{16} = \frac{5}{b + 5}$$
$$4b + 20 = 80$$
$$4b = 60$$
$$\mathbf{b = 15}$$

11. (15)

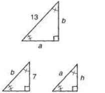


STATEMENTS	REASONS
1. $\overline{BA} \cong \overline{RA}$	1. Given
2. $\overline{AT}$ bisects $\angle BAR$	2. Given
3. $\angle BAT \cong \angle RAT$	3. A bisector divides an angle into two congruent angles.
4. $\overline{AT} \cong \overline{AT}$	4. Reflexive axiom
5. $\Delta BAT \cong \Delta RAT$	5. *SAS* congruency postulate
6. $\overline{BT} \cong \overline{RT}$	6. *CPCTC*

12. (8)

$$\frac{5}{4} = \frac{x}{8}$$
$$4x = 40$$
$$x = \mathbf{10}$$

13. (3)



$$\frac{6}{a} = \frac{a}{13}$$
$$a^2 = 78$$
$$\boldsymbol{a = \sqrt{78}}$$
$$h^2 = \left(\sqrt{78}\right)^2 - 6^2$$
$$h = \sqrt{78 - 36}$$
$$\boldsymbol{h = \sqrt{42}}$$
$$b^2 = 7^2 + \left(\sqrt{42}\right)^2$$
$$b = \sqrt{49 + 42}$$
$$\boldsymbol{b = \sqrt{91}}$$

14. (5)

$$\sqrt{3}\sqrt{5}\sqrt{-3}\sqrt{-5} - \sqrt{3}\sqrt{5}i\sqrt{3}\sqrt{5}i^3 - \sqrt{-36}$$
$$= \sqrt{3}\sqrt{3}\sqrt{5}\sqrt{5}i^2 - \sqrt{3}\sqrt{3}\sqrt{5}\sqrt{5}i^4 - 6i$$
$$= (3)(5)(-1) - (3)(5)(-1)(-1) - 6i = \mathbf{-30 - 6i}$$

15. (13) $x = \dfrac{130 + 150}{2} = \mathbf{140}$

16. (13) $x = \dfrac{40 - 20}{2} = \mathbf{10}$

17. (11,13)

$$3 \cdot x = 6 \cdot 2$$
$$3x = 12$$
$$x = \mathbf{4}$$

18. (13)

$$4(4 + x) = 6(6 + 14)$$
$$16 + 4x = 36 + 84$$
$$4x = 104$$
$$x = \mathbf{26}$$

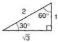
19. (7) **All mathematicians are engineers.** The argument is **valid.** Bobby belongs to the set identified by the contrapositive.

20. (11)

$$\frac{85 + 74 + 91 + 93 + x}{5} = 82$$
$$\frac{343 + x}{5} = 82$$
$$343 + x = 410$$
$$x = \mathbf{67}$$

Test 5

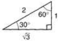
1. (20) (a)



$\tan 30° = \dfrac{1}{\sqrt{3}}$

$$\frac{\sqrt{3}}{2} \tan 30° = \frac{\sqrt{3}}{2}\left(\frac{1}{\sqrt{3}}\right) = \mathbf{\frac{1}{2}}$$

(b)



$\cos 60° = \dfrac{1}{2}$

$$2\sqrt{3} \cos 60° = 2\sqrt{3}\left(\frac{1}{2}\right) = \mathbf{\sqrt{3}}$$

(c)

$\sin 45° = \dfrac{1}{\sqrt{2}}$

$$\frac{\sqrt{2}}{2} \sin 45° = \frac{\sqrt{2}}{2}\left(\frac{1}{\sqrt{2}}\right) = \mathbf{\frac{1}{2}}$$

2. (18)

$$\text{Glycol}_1 + \text{glycol}_2 = \text{total glycol}$$
$$0.79(P_N) + 0.34(72) = 0.63(P_N + 72)$$
$$0.79P_N + 24.48 = 0.63P_N + 45.36$$
$$0.16P_N = 20.88$$
$$P_N = \mathbf{130.50\ liters}$$

Test 11

1. (44) Rate $= \frac{d}{m} \frac{\text{dollars}}{\text{microscope}}$

New rate $= \frac{d}{m} - 20 = \frac{d - 20m}{m} \frac{\text{dollars}}{\text{microscope}}$

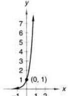
New rate $\times N =$ price

$\left(\frac{d - 20m}{m}\right)N = 2000$

$N = \mathbf{\frac{2000m}{d - 20m}}$ **microscopes**

2. (39) $60°\left(\frac{\pi}{180°}\right)(120) = 40\pi =$ **125.66 in.**

3. (42) $y = \left(\frac{1}{4}\right)^{-2x} = \left[\left(\frac{1}{4}\right)^{-2}\right]^x = 16^x$



4. (41) ${}_nP_r = \frac{n!}{(n - r)!}$

${}_7P_6 - {}_7P_3 = \frac{7!}{(7 - 6)!} - \frac{7!}{(7 - 3)!}$

$= \frac{7!}{1!} - \frac{7!}{4!}$

$=$ **4830**

5. (26,40) (a) $4 \log_2 x = 2 \log_2 16$

$\log_2 x^4 = \log_2 16^2$

$x^4 = 256$

$x^4 = 4^4$

$x =$ **4**

(b) $\log_x 216 = 3$

$x^3 = 216$

$x^3 = 6^3$

$x =$ **6**

(c) $\log_7 \frac{1}{49} = x$

$7^x = \frac{1}{49}$

$7^x = \frac{1}{7^2}$

$7^x = 7^{-2}$

$x =$ **−2**

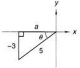
6. (32) (a) $0° \le \theta \le 180°$

Arccos 1 = **0°**

(b) $-90° \le \theta \le 90°$

Arcsin (−1) = **−90°**

(c)



$-90° \le \theta \le 90°$

$a = \sqrt{5^2 - 3^2} = 4$

$\cos\left[\text{Arcsin}\left(-\frac{3}{5}\right)\right] = \mathbf{\frac{4}{5}}$

7. (40) $\log_3 3^6 + \log_4 4^5 - \log_6 6^9 = 6 + 5 - 9 =$ **2**

8. (40) $\log_6 (x + 10) - \log_6 (x - 1) = \log_6 12$

$\log_6 \frac{(x + 10)}{(x - 1)} = \log_6 12$

$\frac{(x + 10)}{(x - 1)} = 12$

$x + 10 = 12x - 12$

$11x = 22$

$x =$ **2**

9. (21) $f(x) = \frac{\sqrt{3 - x}}{x^3 - 2x^2 - 8x} = \frac{\sqrt{3 - x}}{x(x^2 - 2x - 8)}$

$= \frac{\sqrt{3 - x}}{x(x - 4)(x + 2)}$

$x(x - 4)(x + 2) \ne 0$

$x \ne 0, 4, -2$

$3 - x \ge 0$

$x \le 3$

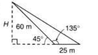
Domain of $f = \mathbf{\{x \in \mathbb{R} \mid x \le 3, x \ne 0, -2\}}$

$$2\cos x + \sqrt{2} = 0$$
$$2\cos x = -\sqrt{2}$$
$$\cos x = -\frac{\sqrt{2}}{2}$$
$$x = \frac{3\pi}{4}, \frac{5\pi}{4}$$
$$x = \frac{\pi}{3}, \mathbf{\frac{3\pi}{4}}, \mathbf{\frac{5\pi}{4}}, \mathbf{\frac{4\pi}{3}}$$

(b) $\tan 2x + \frac{\sqrt{3}}{3} = 0$

$$\tan 2x = -\frac{\sqrt{3}}{3}$$
$$2x = \frac{5\pi}{6}, \frac{11\pi}{6}, \frac{17\pi}{6}, \frac{23\pi}{6}$$
$$x = \mathbf{\frac{5\pi}{12}}, \mathbf{\frac{11\pi}{12}}, \mathbf{\frac{17\pi}{12}}, \mathbf{\frac{23\pi}{12}}$$

20. (56)



$H = 60 \sin 45° = 42.43$ m

Area $= \frac{1}{2}BH = \frac{1}{2}(25)(42.43) = \mathbf{530.38\ m^2}$

Test 18

1. (68) $y = ax^2$

$$y = \frac{1}{28}x^2$$
$$a = \frac{1}{4p}$$
$$\frac{1}{28} = \frac{1}{4p}$$
$$4p = 28$$
$$p = 7$$

The receiver should be placed **7 ft** above the vertex.

2. (68) Vertex: $(h, k) = (-3, 2)$

Focus: $(h, k + p) = (-3, -2)$

$$k + p = -2$$
$$(2) + p = -2$$
$$p = -4$$

Directrix: $y = k - p = 2 - (-4) = 6$

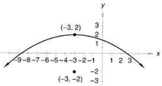
Axis of symmetry: $x = h = -3$

$$y - k = \frac{1}{4p}(x - h)^2$$
$$y - (2) = \frac{1}{4(-4)}[x - (-3)]^2$$
$$y = -\frac{1}{16}(x + 3)^2 + 2$$

Parabola: $\mathbf{y = -\frac{1}{16}(x + 3)^2 + 2}$

Directrix: $\mathbf{y = 6}$

Axis of symmetry: $\mathbf{x = -3}$



3. (67) (a) $\text{antilog}_9 2 = 9^2 = \mathbf{81}$

(b) $\text{antilog}_8 (-2) = 8^{-2} = \frac{1}{8^2} = \mathbf{\frac{1}{64}}$

4. (69)

$$\begin{vmatrix} x + 3 & 2 \\ -3 & x \end{vmatrix} = 10$$
$$x(x + 3) - (-3)(2) = 10$$
$$x^2 + 3x + 6 = 10$$
$$x^2 + 3x - 4 = 0$$
$$(x + 4)(x - 1) = 0$$
$$x = \mathbf{-4, 1}$$

5. (18) $n = \frac{p\sqrt{t}}{m^2}$

$$\frac{2p\sqrt{9t}}{(6m)^2} = \frac{2p(3)\sqrt{t}}{36m^2} = \frac{6p\sqrt{t}}{36m^2} = \frac{1}{6}n$$

***n* is divided by 6.**

6. (64) $(7 \text{ cis } 68°)(-2 \text{ cis } 82°) = -14 \text{ cis } (68° + 82°)$

$= -14 \text{ cis } 150° = -14(\cos 150° + i \sin 150°)$

$= -14\left[\left(-\frac{\sqrt{3}}{2}\right) + i\left(\frac{1}{2}\right)\right] = \mathbf{7\sqrt{3} - 7i}$

7. (1)

$$3(180 - A) = 7(90 - A) + 130$$
$$540 - 3A = 630 - 7A + 130$$
$$4A = 220$$
$$A = \mathbf{55°}$$